FAMILIES INDEX THEOREM IN SUPERSYMMETRIC WZW MODEL AND TWISTED K-THEORY*

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Abstract. The construction of twisted K-theory classes on a compact Lie group is reviewed using the supersymmetric Wess-Zumino-Witten model on a cylinder. The Quillen superconnection is introduced for a family of supercharges and the Chern character for the family is given and its relation to twisted cohomology is discussed.

0. Introduction

Gauge symmetry breaking in quantum field theory is described in terms of families index theory. The Atiyah-Singer index formula gives via the Chern character cohomology classes in the moduli space of gauge connections and of Riemann metrics. In particular, the 2-form part is interpreted as the curvature of the Dirac determinant line bundle, which gives an obstruction to gauge covariant quantization in the path integral formalism. The obstruction depends only on the K-theory class of the family of operators.

In the Hamiltonian quantization odd forms on the moduli space become relevant, [CMM]. The obstruction to gauge covariant quantization comes from the 3-form part of the character. The 3-form is known as the Dixmier-Douady class and is also the (only) characteristic class of a gerbe; this is the higher analogue of the first Chern class (in path integral quantization) classifying complex line bundles.

The next step is to study familes of "operators" which are only projectivly defined; that is, we have families of hamiltonians which are defined locally in the moduli space but which refuse to patch to a globally defined family of operators. The obstruction is given by the Dixmier-Douady class, an element of integral third cohomology of the moduli space. On the overlaps of open sets the operators are related by a conjugation by a projective unitary transformation. This leads to the definition of twisted K-theory.

In the present talk I will review the basic definitions of both ordinary K-theory and twisted K-theory in Section 1. In Section 2 the construction of twisted (equivariant) K-theory classes on compact Lie groups is outlined using a supersymmetric model in $1+1$ dimensional quantum field theory. Finally, in Section 3 the Quillen superconnection formula is applied to the projective family of Fredholm operators giving a Chern character alternatively with values in Deligne cohomology on the base or in global twisted de Rham cocycles, [BCMMS]. The use of Quillen superconnection has been proposed in general context of twisted K-theory in [Fr], but in this talk I will give the details in simple terms using the supersymmetric Wess–Zumino-Witten model.

1. Twist in K-theory by a gerbe class

Let M be a compact manifold and P a principal bundle over M with structure group $PU(H)$, the projective unitary group of a complex Hilbert space H. We shall consider the case when H is infinite dimensional. The characteristic class of P is represented by an element $\Omega \in H^3(M, \mathbb{Z})$, the Dixmier-Douady class.

Choose a open cover $\{U_\alpha\}$ of M with local transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to PU(H)$ of the bundle P.

In the case of a good cover we can even choose lifts $\hat{g}_{\alpha\beta} : U_{\alpha\beta} \to U(H)$, to the unitary group in the Hilbert space H, on the overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$, but then we only have

$$\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha} = f_{\alpha\beta\gamma} \cdot \mathbf{1}$$

for some $f_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to S^1$.

Complex K-theory classes on M may be viewed as homotopy classes of maps $M \to Fred$, to the space of Fredholm operators in an infinite-dimensional complex Hilbert space H. This defines what is known as $K^0(M)$. The other complex K-theory group is $K^1(M)$ and this is defined by replacing $Fred$ by $Fred_*$, the space of self-adjoint Fredholm operators with both positive and negative essential spectrum.

The twisted K-theory classes are here defined as homotopy classes of sections of a fiber bundle Q over M with model fiber equal to either $Fred$ or $Fred_*$. One sets

$$Q = P \times_{PU(H)} Fred,$$

and similarly for $Fred_*$, where the $PU(H)$ action on $Fred$ is simply the conjugation by a unitary transformatio $\hat{g}$ corresponding to $g \in PU(H)$.

We denote by $K^*(M, [\Omega])$ the twisted K theory classes, the twist given by P.

Using local trivializations a section is given by a family of maps $\psi_\alpha : U_\alpha \to Fred$ such that

$$\psi_\beta(x) = \hat{g}_{\alpha\beta}^{-1}(x)\psi_a(x)\hat{g}_{\alpha\beta}(x)$$

on the overlaps $U_{\alpha\beta}$.

2. Supersymmetric construction of $K(M, [\Omega])$

We recall from [M1] the construction the operator Q_A as a sum of a 'free' super-charge Q and an interaction term $\hat{A}$ in (2.7) acting in H. The Hilbert space H is a tensor product of a 'fermionic' Fock space H_f and a 'bosonic' Hilbert space H_b. Let G be a connected, simply connected simple compact Lie group of dimension N and $\mathbf{g}$ its Lie algebra. The space H_b carries an irreducible representation of the loop algebra $L\mathbf{g}$ of level k where the highest weight representations of level k are classified by a finite set of G representations (the basis of Verlinde algebra) on the 'vacuum sector'.

In a Fourier basis the generators of the loop algebra are T_n^a where $n \in \mathbb{Z}$ and $a = 1, .., dim\, G = N$.

The commutation relations are

(2.1)
$$[T_n^a, T_m^b] = \lambda_{abc}T_{n+m}^c + \frac{k}{4}\delta_{ab}\delta_{n,-m},$$

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where the λ_{abc}'s are the structure constant of $\mathbf{g}$; in the case when $\mathbf{g}$ is the Lie algebra of $SU(2)$ the nonzero structure constants are completely antisymmetric and we use the normalization $\lambda_{123} = \frac{1}{\sqrt{2}}$, corresponding to an orthonormal basis with respect to -1 times the Killing form. This means that in this basis the Casimir invariant $C_2 = \sum_{a,b,c} \lambda_{abc} \lambda_{acb}$ takes the value $-N$.

In an unitary representation of the loop group we have the hermiticity relations

$$(T_n^a)^* = -T_{-n}^a$$

With this normalization of the basis, for $G = SU(2)$, k is a nonnegative integer and $2j_0 = 0, 1, 2 \ldots k$ labels the possible irreducible representations of $SU(2)$ on the vacuum sector. The case $k = 0$ corresponds to a trivial representation and we shall assume in the following that k is strictly positive. In general the level k is quantized as an integer x times twice the lenght squared of the longest root with respect to the dual Killing form (this unit is in our normalization equal to 1 in the case $G = SU(2)$); alternatively, we can write $k = 2x/h^\wedge$, where $h^\wedge$ is the dual Coxeter number of the Lie algebra $\mathbf{g}$.

The space H_f carries an irreducible representations of the canonical anticommutation relations (CAR),

(2.2) $$\psi_n^a \psi_m^b + \psi_m^b \psi_n^a = 2\delta_{ab}\delta_{n,-m},$$

and $(\psi_n^a)^* = \psi_{-n}^a$. The representation is fixed by the requirement that there is an irreducible representation of the Clifford algebra $\{\psi_0^a\}$ in a subspace $H_{f,vac}$ such that $\psi_n^a v = 0$ for $n < 0$ and $v \in H_{f,vac}$.

The central extension of the loop algebra at level 2 is represented in H_f through the operators

(2.3) $$K_n^a = -\frac{1}{4} \sum_{b,c;m\in\mathbb{Z}} \lambda_{abc} \psi_{n-m}^b \psi_m^c,$$

which satisfy

(2.4) $$[K_n^a, K_m^b] = \lambda_{abc} K_{n+m}^c + \frac{1}{2} n \delta_{ab}\delta_{n,-m}.$$

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We set $S_n^a = 1 \otimes T_n^a + K_n^a \otimes 1$. This gives a representation of the loop algebra; in the case $G = SU(2)$ the level is $k + 2$ in the tensor product $H = H_f \otimes H_b$. In the parametrization of the level by the integer x this means that we have a level shift $x \mapsto x' = x + h^\wedge$.

Next we define

(2.5)
$$Q = i \sum_{a,n} \left(\psi_n^a T_{-n}^a + \frac{1}{3} \psi_n^a K_{-n}^a \right).$$

This operator satisfies $Q^2 = h$, where h is the hamiltonian of the supersymmetric Wess-Zumino-Witten model,

(2.6)
$$h = -\sum_{a,n} : T_n^a T_{-n}^a : + \frac{k+2}{8} \sum_{a,n} : n\psi_n^a \psi_{-n}^a : + \frac{N}{24},$$

where the normal ordering :: means that the operators with negative Fourier index are placed to the right of the operators with positive index, $: \psi_{-n}^a \psi_n^b : \, = -\psi_n^b \psi_{-n}^a$ if $n > 0$ and $: AB := AB$ otherwise. In the case of the bosonic currents T_n^a the sign is $+$ on the right-hand-side of the equation.

Finally, Q_A is defined as

(2.7)
$$Q_A = Q + i\tilde{k} \sum_{a,n} \psi_n^a A_{-n}^a$$

where the A_n^a's are the Fourier components of the **g**-valued function A in the basis T_n^a and $\tilde{k} = \frac{k+2}{4}$.

The basic property of the family of self-adjoint Fredholm operators Q_A is that it is equivariant with respect to the action of the central extension of the loop group LG. Any element $f \in LG$ is represented by a unitary operator $S(f)$ in H but the phase of $S(f)$ is not uniquely determined. The equivariantness property is

(2.8)
$$S(f^{-1}) Q_A S(f) = Q_{A^f}$$

with $A^f = f^{-1} A f + f^{-1} df$. For the Lie algebra we have the relations

(2.9)
$$[S_n^a, Q_A] = i\tilde{k}(n\psi_n^a + \sum_{b,c;m} \lambda_{abc} \psi_m^b A_{n-m}^c)$$

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The group LG can be viewed as a subgroup of the group $PU(H)$ through the projective representation S. The space $\mathcal{A}$ of smooth vector potentials on the circle is the total space for a principal bundle with fiber $\Omega G \subset LG$, the group of based loops at 1. Since now $\Omega G \subset PU(H)$, $\mathcal{A}$ may be viewed as a reduction of a $PU(H)$ principal bundle over G. The ΩG action by conjugation on the Fredholm operators in H defines an associated fiber bundle $\mathcal{Q}$ over G and the family of operators Q_A defines a section of this bundle. Thus $\{Q_A\}$ is a twisted K-theory class over G where the twist is determined by the level $k+2$ projective representation of LG.

Actually, there is additional gauge symmetry due to constant global gauge transformations. For this reason the construction above leads to elements in the G-equivariant twisted K-theory $K_G^*(G, [\Omega])$, where the G-action on G is the conjugation by group elements. It happens that in the case of $SU(2)$ the construction gives all generators for both equivariant and nonequivariant twisted K-theories, but not for other compact Lie groups.

3. Quillen superconnection

Let Q_A be the supercharge associated to the vector potential A on the circle, with values in the Lie algebra $\mathbf{g}$. Recall that this transforms as

$$\hat{g}^{-1}Q_A\hat{g} = Q_{A^g}$$

with respect to $g \in LG$. Consider the trivial Hilbert bundle over $\mathcal{A}$ with fiber H, the operators Q_A acting in the fibers. Define a covariant differentiation ∇ acting on the sections of the bundle, $\nabla = \delta + \hat{\omega}$ where δ is the exterior differentiation on $\mathcal{A}$ and $\hat{\omega}$ is a connection 1-form defined as follows. First, any vector potential on the circle can be uniquely written as $A = f^{-1}df$ for some smooth function $f : [0, 2\pi] \to G$ such that $f(0) = 1$. A tangent vector at f is then represented by a function $v : [0, 2\pi] \to \mathbf{g}$ such that $v(0) = 0$ with periodic derivatives at the end points, $v = f^{-1}\delta f$. We set

(3.1) $$\omega_f(v) = v - \alpha(x)f(x)^{-1}(\delta f(2\pi)f(2\pi)^{-1})f(x)$$

where α is a fixed smooth real valued function on $[0, 2\pi]$ such that $\alpha(0) = 0, \alpha(2\pi) = 1$ and all derivatives equal to zero at the end points. The meaning of the second

term in (3.1) is that it makes the whole expression periodic so that ω takes values in $L\mathbf{g}$. Then $\hat{\omega}_f(v)$ is defined by the projective representation S of $L\mathbf{g}$ in H.

The gauge transformation $A \mapsto A^g$ corresponds to the right translation $r_g(f) = fg$, which sends $\omega_f(v)$ to $g^{-1}\omega_f(v)g$. However, for the quantized operator $\hat{\omega}$ we get an additional term. This is because of the central extension $\widehat{LG}$ which acts on $\hat{\omega}$ through the adjoint representation. One has

$$\hat{g}^{-1}\hat{\omega}\hat{g} = \widehat{g^{-1}\omega g} + \gamma(\omega, g)$$

with

$$\gamma(\omega, g) = \frac{k+2}{8\pi} \int_{S^1} < \omega, dg g^{-1} >_K .$$

The bracket $< \cdot, \cdot >_K$ is the Killing form on $\mathbf{g}$. But one checks that the modified 1-form

$$\hat{\omega}_c = \hat{\omega} - \frac{k+2}{8\pi} \int_{S^1} < \omega, f^{-1}df >_K$$

transforms in a linear manner,

(3.2) $$\hat{g}^{-1}\hat{\omega}_c\hat{g} = \widehat{r_g\omega_c}.$$

Here r_g denotes the right action of ΩG on $\mathcal{A}$ and the induced right action on connection forms. We would like to construct characteristic classes on the quotient space $\mathcal{A}/\Omega G$ from classes on $\mathcal{A}$ using the equivariantness property (3.2). First, we can construct a Quillen superconnecton [Qu] as the mixed form

(3.3) $$D = \sqrt{t}Q_A + \delta + \hat{\omega}_c - \frac{1}{4\sqrt{t}} < \psi, f >,$$

where f is the Lie algebra valued curvature form computed from the connection ω and $< \psi, f >= \sum \psi_n^a f_{-n}^a$. Formally, this expression is the same as the Bismut-Freed superconnection for families of Dirac operators, [Bi]. Here t is a free positive real scaling parameter. This is introduced since in the case of Bismut-Freed supercon- nection one obtains the Atiyah-Singer families index forms in the limit $t \to 0$ from the formula (3.4) or (3.5) below.

We define a family of closed differential forms on $\mathcal{A}$ from

(3.4) $$\Theta = \mathrm{tr}_s\, e^{-(\sqrt{t}Q_A + \delta + \hat{\omega}_c - \frac{1}{4\sqrt{t}} < \psi, f >)^2}, \quad \dim G \text{ even}$$

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In the case when $\dim G$ is even the supertrace is defined as $\mathrm{tr}_s(\cdot) = \mathrm{tr}\,\Gamma(\cdot)$. Here Γ is the grading operator with eigenvalues ± 1. It is defined uniquely up to a phase ± 1 by the requirement that it anticommutes with each ψ_n^a and commutes with the algebra T_n^a.

To get integral forms the n-form part of Θ should be multiplied by $(1/2\pi i)^{n/2}$. In the odd case the above formula has to be modified:

$$(3.5) \qquad \Theta = \mathrm{tr}^\sigma\, e^{-(\sigma\sqrt{t}Q_A + \delta + \hat\omega_c - \frac{1}{4\sqrt{t}}<\psi,f>)^2},$$

where σ is an odd element, $\sigma^2 = 1$, anticommuting with odd differential forms and commuting with Q_A, and the trace tr^σ extracts the operator trace of the coefficients of the linear term in σ. In this case the n-form part should be multiplied by $\sqrt{2i}(1/2\pi i)^{n/2}$.

The problem with the expressions (3.4) and (3.5) is that they cannot pushed down to the base $G = \mathcal{A}/\Omega G$. The obstruction comes from the transformation property

$$\sqrt{t}Q_{A^g} + \delta + \widehat{r_g\omega}_c - \frac{1}{4\sqrt{t}} < \psi, r_g f >$$

$$(3.6) \qquad = \hat{g}^{-1}(\sqrt{t}Q_A + \delta + \hat\omega_c - \frac{1}{4\sqrt{t}} < \psi, f >)\hat{g} + \hat{g}^*\theta,$$

where θ is the connection 1-form on $\widehat{LG}$ corresponding to the curvature form c on LG, defined by the central extension. Here $\hat{g}$ is a local $\widehat{LG}$ valued function on the base G, implementing a change of local section $G \to \mathcal{A}$. This additional term is the difference

$$\hat{g}^*\theta = \widehat{g^{-1}\delta g} - \hat{g}^{-1}\delta\hat{g},$$

where the first term on the right comes from the tranformation of the connection form $\hat\omega_c$ with respect to a local gauge transformation g. Taking the square of the transformation rule (3.6) we get

$$(3.7) \qquad (r_g D)^2 = \hat{g}^{-1}D^2\hat{g} + g^*c.$$

The last term on the right comes as

$$\delta\hat{g}^*\theta + (\hat{g}^*\theta)^2 = \hat{g}^*\delta\theta = \hat{g}^*c = g^*c,$$

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where in the last step we have used the fact that the curvature of a circle bundle is a globally defined 2-form c on the base, and thus does not depend on the choice of the lift $\hat{g}$ to $\widehat{LG}$.

Theorem. *Let U_α and U_β be two open sets in G with local sections ψ_α, ψ_β to the total space $\mathcal{A}$ of the ΩG principal bundle $\mathcal{A} \to G$. Let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \Omega G$ be the local gauge transformation tranforming ψ_α to ψ_β. Then the pull-back forms Θ_α and Θ_β are related on $U_\alpha \cap U_\beta$ as*

$$\Theta_\beta = \psi_\beta^* \Theta = e^{-g_{\alpha\beta}^* c} \Theta_\alpha.$$

Proof. Since the curvature is closed, $\delta c = 0$, the term $g^* c$ on the right in (3.7) commutes with the rest and therefore can be taken out as a factor $\exp(-g^* c)$ in the exponential of the square of the transformed superconnection.

Remark It is an immediate consequence of the Theorem that the 1-form part $\Theta[1]$ of Θ is a globally defined form on the base G. We can view this as the generalization of the differential of the families η invariant, governing the spectral flow along closed loops in the parameter space; in fact, in the classical case of Bismut-Freed superconnection for families of Dirac operators this is exactly what one gets from the Quillen superconnection formula. We can write $\pi^{-1/2}\Theta[1] = h^{-1} dh/2\pi i$ with $\log h = 2\pi i \eta$. Note that η is only continuous modulo integers. Thinking of η as the spectral asymmetry, we normalize it by setting $\eta(A) = 0$ for the vector potential $A = 0$, or on the base, for the trivial holonomy $g = 1$.

In the odd case we can relate the calculation of $\Theta[3]$ to the computation of the Deligne class in twisted K-theory, [Mi2].

On the overlap $U_{\alpha\beta}$ we have from the Theorem:

$$(2.8) \qquad \Theta_\beta[3] = \Theta_\alpha[3] - g_{\alpha\beta}^* c \wedge \Theta_\alpha[1].$$

This gives

$$(3.9) \qquad \Theta_\alpha[3] - \Theta_\beta[3] = d(\hat{g}_{\alpha\beta}^* \theta \wedge \Theta[1]) \equiv d\omega_{\alpha\beta}^2.$$

Using $\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha} = f_{\alpha\beta\gamma}$ we get

$$(3.10) \qquad \omega_{\alpha\beta}^2 - \omega_{\alpha\gamma}^2 + \omega_{\beta\gamma}^2 = (f_{\alpha\beta\gamma}^{-1} df_{\alpha\beta\gamma}) \wedge \Theta[1].$$

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Choose a function $h : G \to S^1$ as in the Remark, $\pi^{-1/2}\Theta[1] = h^{-1}dh/2\pi i$.

Next the Čech coboundary of the cochain $\{\omega_{\alpha\beta}^2\}$ in (3.10) can be written as

$$(3.11) \qquad\qquad d\omega_{\alpha\beta\gamma}^1 = d(\log(f_{\alpha\beta\gamma})h^{-1}dh).$$

Defining $a_{\alpha\beta\gamma\delta} = \log(f_{\beta\gamma\delta}) - \log(f_{\alpha\gamma\delta}) + \log(f_{\alpha\beta\delta}) - \log(f_{\alpha\beta\gamma})$ we can write

$$(3.12) \qquad\qquad (\partial\omega^1)_{\alpha\beta\gamma\delta} = h^{-a_{\alpha\beta\gamma\delta}}dh^{a_{\alpha\beta\gamma\delta}}$$

where ∂ denotes the Čech coboundary operator. Thus the collection $\{\Theta_\alpha[3], \omega_{\alpha\beta}^2, \omega_{\alpha\beta\gamma}^1, h^{a_{\alpha\beta\gamma\delta}}\}$ defines a Deligne cocycle on the manifold G with respect to the given open covering $\{U_\alpha\}$.

The system of closed local forms obtained from the Chern character formula (3.5) can be modified to a system of *global forms* by multiplication

$$(3.13) \qquad\qquad \Theta' = e^{-\theta_\alpha} \wedge \Theta,$$

where θ_α is the 2-form potential, $d\theta_\alpha = \Omega$ on U_α. One checks easily that now

$$(3.14) \qquad\qquad (d + \Omega)\Theta' = 0.$$

Although the operator $d + \Omega$ squares to zero and can thus be used to define a cohomology theory, [BCMMS], the Chern character should not be viewed to give a map to this twisted cohomology theory. In fact, the twisted cohomology over complex numbers vanishes for simple compact Lie groups. For this reason, in order to hope to get nontrivial information from the Chern character, one should look for a refinement of the twisted cohomology. In fact, there is another integral version of twisted cohomology proposed in [At]. In that version one studies the ordinary integral cohomology modulo the ideal generated by the Dixmier-Douady class Ω. At least in the case of $SU(2)$ it is an experimental fact that the twisted (nonequivariant) K-theory as an abelian group is isomorphic to the twisted cohomology in this latter sense. There is a similar result for other compact Lie groups, but the 3-cohomology class used to define the twisting in cohomology is in general not the gerbe class; both are integral multiples of a basic 3-form, but the coefficients differ, except for the case of $SU(2)$, [Do].

One can explicitly see why the integral cohomology mod Ω is relevant for twisted K-theory by the following construction in the odd case. First we replace the space $Fred_*$ by the homotopy equivalent space $\mathcal{U}_1$ consisting of 1+ trace-class unitaries in H. An ordinary K-theory class on X is a homotopy class of maps $f : X \to \mathcal{U}_1$. In this representation the Chern character defines a sequence of cohomology classes on X by pulling back the generators $\mathrm{tr}\,(g^{-1}dg)^{2n+1}$ of the cohomology of $\mathcal{U}_1$. In the twisted case we have only maps on open sets U_α which are related by $f_\beta = g_{\alpha\beta}^{-1} f_\alpha g_{\alpha\beta}$ on overlaps.

In the case of $G = SU(2) = S^3$ we need only two open sets $U_\pm$, the slightly extended upper and lower hemispheres, and a map g_{-+} on the overlap to the group $PU(H)$, of degree k. If now $f_- = 1$ identically and the support of f_+ is concentrated at the North pole, then the pair $f_\pm$ is related by the conjugation by $g = g_{-+}$ at the equator and at the same time it defines an ordinary K-theory class since the functions patch to a globally defined function on S^3. Let us assume that the winding number of $f : S^3 \to \mathcal{U}_1$ is k. Next form a continuous path $f_\pm(t)$ of representatives of twisted K-theory classes starting from $f(1) = f$ and ending at the trivial class represented by the constant function $f(0) = 1$. Let ρ be a smooth function on S^3 which is equal to 1 on the overlap U_{-+} and zero in small open neighborhoods $V_\pm$ of the poles. We can also extend the domain of definition of $P(x)$ to a larger set $U_+ \setminus V_+$. For $0 \leq t \leq 1$ define

$$f_-(t) = e^{2\pi it\rho(x)P_0} \text{ and } f_+(t) = e^{2\pi it\rho(x)P(x)}$$

where $P(x) = g^{-1}(x)P_0 g(x)$, with P_0 is a fixed rank one projection, $x \in U_{-+}$. These are smooth functions on $U_\pm$ respectively and are related by the conjugation by g on the overlap. But for $t = 0$ both are equal to the identity $1 \in \mathcal{U}_1$. On the other hand, at $t = 1$ the integral

$$\frac{1}{24\pi^2} \int_{S^3} \mathrm{tr}\,(f^{-1}df)^3$$

is easily computed to give the value k. This paradox is explained by the fact that for the intermediate values $0 < t < 1$ the functions $f_\pm$ do not patch up to a global function on S^3. Thus we have a homotopy joining a pair of (trivial) twisted K-theory

classes corresponding to the pair of third cohomology classes $0, \Omega$ computed from the Chern character. This confirms the claim, at least in the case of $X = S^3$, that the values of the Chern character should be projected to the quotient $H^*(X, \mathbb{Z})/\Omega$.

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